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LETTER TO THE EDITOR

Alternative Hamiltonian quantization

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Abstract. We take a standard Hamiltonian, quadratic in a set of creation and annihilation operators, and demonstrate an alternative quantization by quommutators; i.e. a set of rules for transposition of these operators, different from the those of the usual canonical commutation relations which yields a new spectrum, of a type reminiscent of logarithmic trajectories.

There is considerable activity at present in the issue of the utility of quantum groups in physics and the question as to whether anything new may be obtained by their use which is more than just a reformulation of known results. From our point of view, the work of Curtright and Zachos [1], in which they exhibit explicit formulas for the representations of q-deformed Lie algebras in terms of the undeformed ones, creates the suspicion that a deformed theory can always be written in terms of variables obeying standard canonical commutation relations, and is merely a (probably complicated) transformation of a known tractable case. This suspicion was confirmed [2] by the construction of transformations which directly represent examples of deformed canonical commutation relations in terms of standard Fock space operators. Similar observations about the utility of quantum groups have been made by other workers [3].

On the other hand, the existence of such transformations notwithstanding, it turns out to be instructive to take a solvable Hamiltonian, and attempt to quantize it using deformed commutators. An example of this sort in a spin–chain system has been shown to possess $SU(2)_q$ symmetry [4]. We shall demonstrate a simple example of an alternative quantization of a standard quadratic Hamiltonian such that the spectrum is different from the one obtained by canonical quantization. Our solution is multiparametric, and though set out as finite dimensional, it could easily be generalized to the infinite-dimensional (field theoretic) situation and prove interesting as the spectrum posesses quite different properties from the usual case.

This will be achieved by using quommutators rather than the usual commutators or anticommutators. As is well known, the Jacobi identities which have to be satisfied in the usual case are replaced by the associativity requirements on the triple products of the operators

$$(a*b)*c=a*(b*c). \tag{1}$$

There exists a considerable literature on the deformation of a single harmonic oscillator Hamiltonian [5]. Our immediate antecedents are in the papers of Macfarlane [6] and Beidenharn [7], of which our results are a non-trivial generalization to any number of oscillators.

We consider a set of rules for a set of N operators $a^\dagger(j)$, $0<j\leq N$, which are considered as creation operators with an equal number of destruction operators $a(j)$ which annihilate the vacuum state. We also introduce $a(0)$ as an extra operator. The total set of $2N+1$ operators $a(j)$, $a^\dagger(j)$ and $a(0)$ can then be put in correspondence with the set $x(j)$ where x's with positive j correspond to the creation operators, the x's with negative j to the annihilation operators, and $x(0)$ to $a(0)$.

We extend the notion of normal ordering by requiring:

(i) that destruction operators appear on the right of creation operators;

(ii) that a product of creation operators $a^\dagger(j_1)a^\dagger(j_2)\ldots a^\dagger(j_n)$, with $0\leq j_i\leq N$, is written in decreasing order of its indices;

(iii) that a product of annihilation operators $a(j_1)a(j_2)\ldots a(j_n)$, with $0\leq j_i\leq N$, is written in increasing order of its indices.

In terms of the x's this means that:

(iv) in the normal products of x's the indices are in decreasing order.

The rules we propose implement associativity of this product, and guarantee that all ways of re-ordering a given product to put it in normal form are equivalent. We have found a very general parametrization under the hypothesis of quommutators and 'restricted grading' which can be summarized as follows. In particular they guarantee that the sum of the indices in the x's be preserved under reordering.

$$
\begin{aligned}
&a^\dagger(j)a^\dagger(k)=r_{jk}a^\dagger(k)a^\dagger(j) && \text{for } N\geq k>j\geq 1\\
&a(j)a(k)=c_{jk}a(k)a(j) && \text{for } N\geq j>k\geq 1\\
&a(j)a^\dagger(k)=s_{jk}a^\dagger(k)a(j) && \text{for } j>0,\ k>0 \text{ and } j\neq k\\
&a(j)a^\dagger(j)=\sum_{k=1}^{j} f_{jk}a^\dagger(k)a(k)+d_j && \\
&a(0)a^\dagger(j)=r_{0j}a^\dagger(j)a(0) && \text{for } N\geq j\geq 1\\
&a(j)a(0)=c_{j0}a(0)a(j) && \text{for } N\geq j\geq 1.
\end{aligned}
\tag{2}
$$

It is obvious that, if they are non-zero, the parameters d_j can be renormalized to 1 by rescaling of the creation and/or annihilation operators.

Under rather general conditions the restrictions imposed by the associativity requirements are as follows:

$$s_{jk}=\frac{1}{r_{jk}} \qquad \text{for } 1\leq j<k\leq N \tag{3}$$

$$s_{jk}=\frac{1}{c_{jk}} \qquad \text{for } N\geq j>k\geq 1 \tag{4}$$

$$c_{a0}=r_{0a} \qquad \text{for } 1\leq a\leq N. \tag{5}$$

In order to write the remaining restrictions let us also define the quantities m_{jk} by

$$m_{jk}=\frac{1}{r_{jk}c_{kj}} \qquad \text{for } 1\leq j<k\leq N. \tag{6}$$

Then m_{jk} is forced to depend only on its first index

$$m_{jk}=\lambda_j \qquad \text{for } 1\leq j<k\leq N. \tag{7}$$

This equation defines λ_j for j up to $N-1$. It is then convenient to introduce f_{NN} as λ_N. The parameters f_{jk} are then given by

$$\begin{aligned} f_{jk} &= \frac{\mathrm{d}_j}{\mathrm{d}_k}(\lambda_k - 1) && \text{for } N \geqslant j > k \geqslant 1 \\ f_{jj} &= \lambda_j && \text{for } 1 \leqslant j \leqslant N. \end{aligned} \tag{8}$$

The parameters r_{jk} and c_{kj}, defined for $k>j>0$ may be regarded as the superdiagonal and the subdiagonal entries in the same $N \times N$ matrix. There are $(N-1)(N-2)/2$ relations between them which prescribe through (7) all c_{kj} in terms of r_{jk} and of the $N-1$ free parameters λ_a for $1 \leqslant a \leqslant N-1$.

Let us finally note that if the adjoint operators are defined in the usual way in the usual Fock space, the parameters are restricted to satisfy

$$c_{kj} = r_{jk}^*. \tag{9}$$

This in turn implies that the λ's are real positive and that the modulus of r_{ij} depends on its first index only.

Now consider the Hamiltonian (quadratic up to the last term):

$$H = \sum_{j=1}^{N} h_j a^\dagger(j) a(j) + h_0 a(0)^2 + h_s a(0). \tag{10}$$

We then solve the problem of obtaining a secular equation of the following form for an operator X being a linear combination of the a's and $a^\dagger$'s:

$$HX - \lambda X H = \rho X. \tag{11}$$

This equation is a quommutator generalization of the usual commutation relations between the Hamiltonian and the creation and annihilation operators. The algebra itself is an extension of that proposed by Vokos, Zumino, Wess and Schirrmacher [8–10] in their development of a differential calculus on quantum planes.

After some algebra one obtains the following results for the allowed Hamiltonian,

$$H = \sum_{j=1}^{N} \frac{(\lambda_j - 1)}{d_j} a^\dagger(j) a(j) + h_0 a(0)^2 + h_s a(0) \tag{12}$$

where the product $h_0 h_s$ has to vanish

$$h_0 h_s = 0. \tag{13}$$

When $h_0 = 0$ the extra restrictions on the parameters are

$$r_{0j} = \lambda_j \qquad \text{for } N \geqslant j \geqslant 1 \tag{14}$$

while when $h_s = 0$ the restrictions are

$$r_{0j} 2 = \lambda_j \qquad \text{for } N \geqslant j \geqslant 1. \tag{15}$$

The solutions X`of (11) are the $x(j)$ and their corresponding eigenvalues are

$$\begin{aligned} x(j) &= a^\dagger(j) && \lambda = \lambda_j && \rho = \lambda_j - 1 \\ x(-j) &= a(j) && \lambda = \frac{1}{\lambda_j} && \rho = -\frac{\lambda_j - 1}{\lambda_j} \\ x(0) &= a(0) && \lambda = \lambda_0 = 1 && \rho = 0. \end{aligned} \tag{16}$$

For a multiparticle state

$$|\psi\rangle = \prod_{j=0}^{N} a^{\dagger}(j) n_j |0\rangle \tag{17}$$

with the usual quantization scheme, the spectrum of the Hamiltonian (12) is simply

$$E = \sum_{j=1}^{N} n_j(\lambda_j - 1). \tag{18}$$

In the case when the quantization is done with the associative rules of our quommutators, we obtain

$$E = \prod_{j} \lambda_j^{n_j} - 1 \tag{19}$$

instead.

The spectrum we have obtained (19) is reminiscent of what has once been called logarithmic trajectories [11-13] and may well be the natural way to implement these type of trajectories in a quantum field theory. There may also be an application to the theory of squeezed states in quantum optics; the single oscillator deformation has already appeared in this context [14].

Needless to say, this approach poses many difficult problems. We here cite a few.

(1) We have tried to extend this to an infinite qualgebra of the Virasoro type, as the Hamiltonian (12) is a natural candidate for L_0 in the infinite limit, but the obvious generalization to L_n fails.

(2) It would be interesting to try to obtain a Lagrangian approach to this theory, and to construct a quantum mechanics evolving in time

(3) Lorentz invariance should be introduced in some way. Does $a(0)$ play a special role in this respect?

(4) What happens in the presence of interactions? What further restrictions are necessary so that the creation and annihilation operators can be regarded as Fourier coefficients in the expansion of a field, and how does one develop a perturbation theory? Should the Hamiltonian be modified? There is the tantalizing prospect that in the particular case when the interactions are of exponential type, so that the theory is an affine, or conformal Toda theory, a quantization exists for the theory as a whole, without resort to a perturbative procedure.

(5) Operator representations of the algebra can probably be constructed.

We hope to come back later to some of these problems.

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